UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 4th FEBRUARY 2010

Organised by the **United Kingdom Mathematics Trust from the School of Mathematics, University of Leeds**

http://www.ukmt.org.uk

The Actuarial Profession
making financial sense of the future

SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

The UKMT is a registered charity

1. D $10 + 10 \times 10 \times (10 + 10) = 10 + 10 \times 10 \times 20 = 10 + 2000 = 2010$.

2. A The sum of the interior angles of a quadrilateral is 360°, so the fourth angle is $(360 - 3 \times 80)° = 120°$.

3. E 2345 has units digit 5 and so is a multiple of 5; 23 456 is even; the digit sum of 234 567 is 27 so it is a multiple of 9; 2 345 678 is even. So if exactly one of the numbers is prime then it must be 23 456 789.

4. A The number of calories saved per day is $\frac{7000}{365} \approx \frac{7000}{350} = 20$.

5. E The values are A $\frac{1}{50}$, B $\frac{1}{60}$, C $\frac{1}{60}$, D $\frac{1}{50}$, E $\frac{1}{30}$.

6. C Triangle PQS is isosceles with $PS = QS$ so $\angle PQS = \angle SPQ = 20°$. Therefore $\angle PSR = 20° + 20° = 40°$ (exterior angle theorem). Triangle PSR is also isosceles, with $PS = PR$, so $\angle PRS = \angle PSR = 40°$.

7. D The Festival will next be held in Worcester in 2011. As it follows a three-year cycle, the Festival is held in Worcester when the number of the year leaves a remainder of 1 when divided by 3. So it will be held in Worcester in 2020, 2032, 2047 and 2077, but not in 2054.

8. B The next such display will be 03:12, that is in 41 minutes' time.

9. C The difference in perimeters is the total length of the edges which are hidden when the pieces are fitted together. These are eight straight edges of length 1 and four semicircular arcs of radius 1.
So the required difference is $8 \times 1 + 4\left(\frac{1}{2} \times 2 \times \pi \times 1\right) = 8 + 4\pi$.

10. C Every year, the day of the week on which a particular date falls is one day later than it fell the previous year unless February 29th has occurred in the meantime, in which case it falls two days later. As January 1st returned to a Monday after 11 years, it must have ‘moved on’ 14 days during that time, so February 29th occurred three times in those 11 years.

11. B If the first statement is true, then the three other statements are all false. If the first statement is false, however, then the second statement is the only true statement. Either way, exactly one of the four statements is true.

12. D When the cuboid is cut away, the surface area of the solid ‘loses’ two rectangles measuring 10 cm × 5 cm and two squares of side 5 cm. However, it also ‘gains’ two rectangles measuring 10 cm × 5 cm. So the surface area decreases by an area equal to one half of the area of one of the faces of the original cube, that is one twelfth of its original surface area.

13. B Let the prices of a fork handle and a candle be £x and £y respectively.
Then $x + y = 6.1$ and $x - y = 4.6$. Adding these two equations gives $2x = 10.7$.
So a fork handle costs £5.35 and a candle costs £0.75.
Therefore the required total is £10.70 + £3.00 = £13.70.

14. C Adding the three equations gives $3x + 3y + 3z = 30$, so $x + y + z = 10$.
(*The equations may be solved to obtain* $x = 2, y = 3, z = 5$. *However, as the above method shows, this is not necessary in order to find the value of* $x + y + z$.)

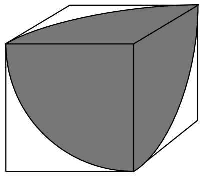
15. E The line $y = 2x + 6$ intersects the y-axis when $x = 0$ and $y = 6$. It intersects the x-axis when $x = -3$ and $y = 0$. So E is the correct line.
(*Alternatively*: $y = 2x + 6$ may be rearranged to give $x = \frac{1}{2}y - 3$. So the required line looks the same as the line $y = \frac{1}{2}x - 3$ when the axes are drawn in the traditional way.)

16. E After x hours, the first clock will have gone forward $2x$ hours and the second clock will have gone back x hours. So the next time they agree is when $2x + x = 24$, that is when $x = 8$. The correct time then is 21:00.

17. A The volume of a cylinder of radius r and height h is $\pi r^2 h$. Replacing r by $2r$ and h by $3h$ multiplies this volume by 12.

18. A Let the distance from the chalet to the top of the mountain be x miles. Then, at 6 mph Supergran would take $\frac{x}{6}$ hours, whereas at 10 mph she would take $\frac{x}{10}$ hours. So $\frac{x}{6} - \frac{x}{10} = 2$, that is $5x - 3x = 60$, so $x = 30$. Hence Supergran's departure time is 8 am and to arrive at 12 noon she should walk at $\frac{30}{4}$ mph, that is $7\frac{1}{2}$ mph.

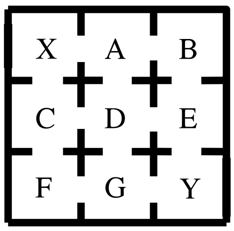
19. B In one hour, the snail can reach points within 1 m of the corner at which it starts. So it can reach some of the points on the three faces which meet at that corner, but none of the points on the other three faces. On each of the three reachable faces, the points which the snail can reach form a quarter of a circle of radius 1 m.



So the required fraction is $\dfrac{3 \times \frac{1}{4}\pi \times 1 \times 1}{6 \times 1 \times 1} = \dfrac{\pi}{8}$.

20. D If the difference between $\sqrt{n}$ and 7 is less than 1, then $6 < \sqrt{n} < 8$. Therefore $36 < n < 64$, so there are 27 possible values of n.

21. E The rooms are labelled A, B, C, D, E, F, G, X, Y as shown. We look first at routes which visit no room more than once. We need consider only routes which go from X to A, since each of these routes has a corresponding route which goes from X to C. For example, the route X A D E Y corresponds to the route X C D G Y.

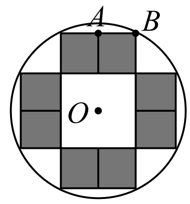


Routes which start X A then go to B or to D. There are three routes which start X A B, namely X A B E Y, X A B E D G Y and X A B E D C F G Y. There are also three routes which start X A D, namely X A D E Y, X A D G Y and X A D C F G Y.
The condition that a gap in a wall closes once a person has travelled through it means that it is not possible to visit a room more than once unless that room has at least four gaps leading into and out of it and the only such room is D. There

are two routes which start X A and visit D twice. These are X A D G F C D E Y and X A D C F G D E Y. So there are 8 routes which start X A and there are 8 corresponding routes which start X C so there are 16 routes in all.

22. E Curly's drink has squash and water in the ratio 1: 7, whilst the corresponding ratio for Larry's drink is 3 : 37. This ratio is less than 1 : 7. When some of Curly's mixture is poured into Larry's, the strength will be between 1 : 7 and 3 : 37, but not equal to either.

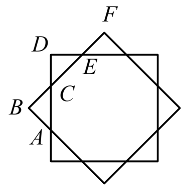
23. B Let the centre of the circle be O and let A and B be corners of one of the shaded squares, as shown. As the circle has area π units², its radius is 1 unit. So OB is 1 unit long. Let the length of the side of each of the shaded squares be x units.



By Pythagoras' Theorem: $OB^2 = OA^2 + AB^2$, that is $1^2 = (2x)^2 + x^2$.
So $5x^2 = 1$. Now the total shaded area is $8x^2 = 8 \times \frac{1}{5} = 1\frac{3}{5}$ units².

24. B There is the possibility of using only 3s giving one possible number 333333. Let's suppose a second digit is used, say x. After the initial digit 3, there are 5 positions into which we can put either 3 or x. So there are 2 choices in each of these 5 positions and so $2^5 = 32$ possible choices − except that one such choice would be five 3s. So we get 31 choices. There are 9 possible values for x, namely 0, 1, 2, 4, 5, 6, 7, 8, 9. So this gives $9 \times 31 = 279$ numbers. Together with 333333, this gives 280 numbers.

25. D Let the length of the side of regular octagon be x units and let A, B, C, D, E, F be the points shown. So $AC = CE = x$. Now $\angle ACE = 135°$ (interior angle of regular octagon), so $\angle ACB = 45°$ and hence triangle ABC is an isosceles right-angled triangle with $AB = BC$.



Also, by Pythagoras' Theorem: $AB^2 + BC^2 = AC^2 = x^2$ so $AB = BC = \frac{\sqrt{2}}{2}x$.
Similarly, $EF = \frac{\sqrt{2}}{2}x$.
Therefore $BF = \left(\frac{\sqrt{2}}{2}x + x + \frac{\sqrt{2}}{2}x\right)$ units $= x\left(1 + \sqrt{2}\right)$ units.
But we are given that $BF = \left(1 + \sqrt{2}\right)$ units so $x = 1$.
Now the area of the octagon formed by the overlap of the squares is equal to the area of one of these squares minus the sum of the area of four triangles, each of which is congruent to triangle CDE.
Thus, in square units, the required area is

$$\left(1 + \sqrt{2}\right)^2 - 4 \times \frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = 3 + 2\sqrt{2} - 1 = 2 + 2\sqrt{2}.$$